Abstract

The perspective projection of a spheroid of known dimensions onto an image plane is derived. This paper first considers the projection of a 2-D ellipse projected onto an image line and then applies the results to the 3-D scenario. An elegant set of algebraic expressions for the ellipse parameters are derived in terms of the spheroid parameters, followed by a demonstration and discussion on their interpretation. In this paper we present a generalization of homogeneous coordinates that enables us to express the perspective projection of a spheroid onto an image plane in a more readily accessible way.

1 Introduction

The projections of simple geometric objects and polygons onto planes is not a new topic and much literature has been gathered over the years for such scenarios. In [1] Coxeter starts by defining projective geometry before considering projections of triangles, quadrangles and 2D conics. In [2] Casey continues this study from a more analytical standpoint. Semple and Kneebone consider simpler projection models in [3] and Duda and Hart cover perspective transformations and invariants in [4]. Typically a simple type of projection is considered - the orthographic projection (see Millar and Maclin, [5]): given a coordinate system \(x, y, z\) and a ‘focal plane’ existing on \(x = f\) with normal vector \((1, 0, 0)\), the orthographic projection onto this plane is defined as the mapping of all points \((x, y, z)\) to \((f, y, z)\). Essentially it is a mapping where all points in space are projected or collapsed vertically onto a focal plane - see figure (1).

This approximation is suitable for some scenarios, where the depth of the object of interest is small in comparison to the object’s distance from the camera. However a typical pin-hole camera is more accurately described mathematically by a perspective projection where the focal plane is replaced by a focal point and the concept of an image plane at
$x = f$ is used: all points $(x, y, z)$ are projected towards this focal point. In this paper the focal point is positioned at the origin of a camera coordinate system. The intersection of these projections onto the image plane gives the captured picture. The camera frame and perspective projection is illustrated in figure (1).

There are many applications which use (or would like to use) the projections and reconstructions of these objects or geometric representations, for example in medical imaging. Jaggi, Karl and Willsky [6] explain the desire to reconstruct an $n$-dimensional dynamically evolving ellipsoid in order to process myocardial perfusion images; the ejection fraction (the fraction of blood pumped out of a ventricle with each heart beat) is estimated by treating the ventricle as an ellipsoid and finding its size based on a collection of 2D images. Because of a current gap in a clear reconstruction analysis Jaggi et al. assume that the images found are aligned with the ellipsoid axes and hence introduce distortions in the ventricle model. In robotics Roberts [7] describes methods for reconstructing planar-surfaced objects from their perspective projections which provides a valuable starting point for future object detection systems but has not developed the theory to further objects such as conics. In motion planning Poignet et al. and Liu et al. ([8],[9]) describe how ellipsoids are often used to describe an object’s positional error in order to calculate collision risk. From its projection an ellipsoid can only be reconstructed up to a single parameter however modeling the target’s error in position with a spheroid yields unique solutions which could then mean faster image processing and collision risk analysis. Similarly in ray tracing Bouville describes how modeling an object as a conic lends faster calculation times to image rendering based on their projections and an analytic algorithm would be more efficient still [10].

Because the mathematical construct of a perspective projection is much more complicated (as described by Zisserman and Mundy in [11]) this approach is typically avoided and estimations are settled with. Further approximative descriptions and methods are described by Horrau et al. in [12] with varying levels of accuracy. Two examples of these alternative projections are the strong-perspective and weak-perspective projection models which combine both the orthographic and the perspective projection to give a more accurate approximation of the perspective projection than the orthographic projection (described by Osterman et al. in [13]).

There exist analytic and numerical methods for true perspective projections of certain geometric objects, however no literature exists for a spheroid. A spheroid is defined as a degenerate ellipsoid: there exists an axis of symmetry for the spheroid. In [14] Eberly gives a most elegant method for finding the numerical perspective projection of an ellipsoid: by maintaining the quadratic form of the ellipsoid Eberly uses a combination of geometric and algebraic arguments to quickly determine the ellipsoid projection. This approach, however, does not give a simplified description of the resulting ellipse as a function of the ellipsoid parameters. It is therefore difficult to gain a geometric understanding of the projection and how it is dependent upon the parameters of the ellipsoid. Traditionally, the perspective projection of geometric objects has been treated using homogeneous coordinates [15]. This approach introduces another dimension to the problem and the reduction of dimensionality to real coordinates is achieved by considering ratios of homogeneous coordinates (see Wylie [15], Springer [16] and Semple & Kneebone [17]). The projection of an ellipsoid or spheroid proceeds through a series of stages. First define a polar plane which is then projected onto the image plane. The intersection of the polar plane and the ellipsoid determines a set of
points which project onto the image ellipse. Hence it is possible, to relate the parameters of the image ellipse to those of the ellipsoid.

In our analysis we consider the perspective projection of a spheroid. We derive an alternative way of relating the spheroid parameters to those of its image ellipse, which avoids defining the polar plane. The complexity of this algorithm is comparable to that using homogeneous coordinates, and provides a simple geometric understanding. A key aspect of this approach is that we show there exists a generalization of homogeneous coordinates in a higher dimensional space where a single point represents a complete spheroid. These generalized homogeneous coordinates then directly relate the spheroid and image ellipse parameters.

This paper derives analytically the perspective projection of a spheroid onto an image plane by utilizing it’s symmetry and geometric properties. The process for doing this starts with a lower dimensional equivalent problem of the projection of an ellipse onto a line. In section (3) results of the 2D projection are examined and the perspective distortions are highlighted. Section (4) defines the frames and structure of the problem before calculating the intersection of a spheroid with a plane. This intersection is an ellipse and it is this result along with those gained in section (2) that enables us to reach an algebraic description of the projected spheroid onto the image plane (described in section (5)). Then section (6) looks to reduce these equations to a simpler and more understandable set of expressions. Results and examples of the derived expressions are shown in section (7) which are then discussed in section (8).
2 Perspective Projection of an Ellipse onto a Line

The lower dimensional scenario of the perspective projection of an ellipse onto an image line is considered. This has specific applications alone (for example in analyzing CT scans Theodorou et al. ([18]) describe arteriovenous malformations as a prism made of adjacent slices with elliptical shapes) however the results are also used in finding the perspective projection of a spheroid, as seen later on.

In a coordinate system with the origin placed at the centre of an ellipse and with axes oriented with the principal axes of that ellipse, that ellipse is defined (see Casey, [2]) as the set of points \((\xi, \eta) \in \mathbb{R}^2\) satisfying

\[
\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1
\]  

for fixed, positive, real values \(a\) and \(b\). The semi-major axis is then defined as \(a\) and the semi-minor axis as \(b\). The ellipse can then be rotated by \(\theta\) and its centre translated to \((x_c, y_c)\). This then describes an ellipse in the camera frame with semi-major axis \(a\), semi-minor axis \(b\), orientation \(\theta\) with respect to the \(x\)-axis and centre \((x_c, y_c)\), where the camera frame has its origin centred on the focal point of the projection and its \(x\)-axis aligned with the focal axis.

We state the equivalent lower-dimensional scenario: Given an ellipse on the \(x-y\) plane with known centre \((x_c, y_c)\), orientation \(\theta\) and Semi-axes \(a\) and \(b\), how is the ellipse perspective projected towards the origin onto the line \(x = 1\)?

By fixing the ‘image line’ to be at a unit distance we effectively normalise the distances of this 2D scenario. Any scaling of the target size and distance from the origin is now controlled through the image line distance. See figure (2) for a diagrammatic representation of the problem.

The projection of the ellipse onto the image line is given by a chord on the \(y\)-axis. This chord is defined by its maximum and minimum values \(m_+\) and \(m_-\). The two lines \(y = m_+x\) and \(y = m_-x\) will touch the ellipse once, lying tangent to the edge.

Consider the ellipse frame and define new axes \((\xi, \eta)\) as shown in figure (2).

The two coordinate frames are related by the following equations:

\[
\begin{align*}
  x - x_c &= \xi \cos \theta - \eta \sin \theta \\
  y - y_c &= \xi \sin \theta + \eta \cos \theta
\end{align*}
\]  

(2)

We can now re-write the equations for the bounding line in terms of the new coordinates:

\[
\xi (\sin \theta - m_\pm \cos \theta) + \eta (\cos \theta + m_\pm \sin \theta) + (y_c - m_\pm x_c) = 0
\]  

(3)

where \(m_\pm\) denotes either \(m_+\) or \(m_-\). Now consider the ellipse alone. Any point on the ellipse satisfies equation (1) and we can parameterise the ellipse using a parameter \(t\), such that

\[
\xi = a \cos t \quad \eta = b \sin t
\]  

(4)

When we express just one power of \(\xi\) and \(\eta\) in equation (1) in terms of the new parameter we come to:

\[
\xi b \cos t + \eta a \sin t - ab = 0
\]  

(5)
3 Ellipse Perspective Projection Results

The perspective projection of an ellipse onto an image line is relatively straightforward to calculate and to visualize. However, subtleties exist in the 3D spheroid projection which

Figure 2: Illustrating the configuration for an ellipse projection onto the line \( x = 1 \) and the ellipse axes.

This is an equation for a line which is tangent to the ellipse at the point \( t \). Therefore if we wish the gradient line given by (3) to be this same line (and therefore the bounding line, as it passes through the origin and is tangent to the ellipse), then for some given \( t \approx t' \) equations (3) and (5) are the same for all \( \xi \) and \( \eta \), up to a scalar multiple. To remove the scalar ambiguity we divide the coefficients to obtain

\[
\frac{b \cos t'}{\sin \theta - m_{\pm} \cos \theta} = \frac{a \sin t'}{\cos \theta + m_{\pm} \sin \theta} = \frac{ab}{m_{\pm} x - y}
\]  

These equations can be re-arranged to eliminate \( t' \):

\[
(m_{\pm} x - y)^2 = a^2(\sin \theta - m_{\pm} \cos \theta)^2 + b^2(\cos \theta + m_{\pm} \sin \theta)^2
\]  

Solving for \( m_{\pm} \) in terms of the ellipse parameters obtains:

\[
m_{\pm} = \frac{2x y - (a^2 - b^2) \sin 2\theta \pm 2\sqrt{\Delta}}{2x^2 - (a^2 + b^2) - (a^2 - b^2) \cos 2\theta}
\]  

where

\[
\Delta = \frac{1}{2}(x^2 + y^2)(a^2 + b^2) - \frac{1}{2}(x^2 - y^2)(a^2 - b^2) \cos 2\theta - x y (a^2 - b^2) \sin 2\theta - a^2 b^2
\]  

This equation we will later refer to as the 2D \( m_{\pm} \) function.
is best visualized by considering the 2D equivalent scenario. In particular the effect of perspective stretching can be seen for the lower dimensional case as illustrated here.

Figure (3) shows a typical perspective projection of an ellipse onto the image line. In this scenario $y_c = 0$ so that the ellipse lies on the focal axis. One might expect that the two bounding lines given by $m_+$ and $m_-$ would then be symmetric about the $x$ axis. However, figure (3) illustrates that in general this is not the case: the points which lie on the intersection of the ellipse and the bounding lines are not symmetrically placed about the ellipse centre - this is a result of the perspective projection. As such the only time this is true is when $\theta = 0^\circ$ or $90^\circ$.

![Figure 3: Single projection of an ellipse onto the image line: $a = 4$, $b = 2$, $x_c = 10$, $y_c = 0$, $\theta = 20^\circ$](image)

If we fix the ellipse size and centre position but then rotate the ellipse around we can see how this shift in the line projection moves by considering the average of $m_+$ and $m_-$. The variation with respect to $\theta$ is shown in figure (4).

Figure (5) looks at the same system: a fixed centre of $x_c = 5$, $y_c = 0$, with $a = 4$ and $b = 2$ and changing $\theta$. In this graph the specific curves of $m_+$ and $m_-$ are drawn against $\theta$. The non-symmetric behaviour induced by the perspective projection lessens with by increasing $x_c$; as $x_c$ increases the perspective projection tends towards the orthographic projection. However note that we still have $y_c = 0$. If we consider a scenario with $y_c \neq 0$ (figures (6) and (7)) a similar appearance to the curves in figure (5) is found, again due to the perspective nature of the projection. As expected and clarified in figure (6) these two curves are bounded by the $m_{\pm}$ values which would be given for circles of radii $a$ and $b$. 
4 INTERSECTION OF A SPHEROID WITH A PLANE

The goal of this section is to find the intersecting curve of the spheroid with a plane that contains the point $X_D$ and has normal vector $n$, as shown in figure (8). The projection of a spheroid can be calculated by finding the intersection of a spheroid with a plane, projecting that curve onto a line $l$ and then rotating the plane about a vector orthogonal to that line to sweep out $\mathbb{R}^3$. That vector is the focal axis of the camera frame and the image plane is described by the plane swept out by the line $l$. The initial stage is deriving the intersection between a spheroid and a plane, which we now derive.

In a coordinate system with the origin placed at the centre of a spheroid and with axes oriented along that spheroid’s principal axes, then that spheroid is defined (see Tietze, [19]) as the set of points $(u, v, w) \in \mathbb{R}^3$ satisfying

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} + \frac{w^2}{B^2} = 1 \quad (10)$$

for fixed, positive values $A$ and $B$. The axis of symmetry has half-length $A$ and the other two axes have half-lengths $B$. If $A > B$ and the semi-major axis is along the axis of symmetry then the spheroid is said to be prolate. Conversely if $B > A$ then the spheroid is said to be oblate. The spheroid can then be rotated so that the axis of symmetry points in a direction $p$ and its centre translated by a vector $q$.

Create a spheroid whose half-length along the axis of symmetry is $A$ and half-length along the other axes $B$, to be centred at the origin of a $(u, v, w)$ coordinate system, with the axis of symmetry pointing along the $u$-axis. We refer to this frame as the spheroid frame. Define $n$ to be a unit vector and a general point $u = (u, v, w)$.

Figure 4: Demonstrating the variation in the centre of the projection of an ellipse onto the image line: $a = 4$, $b = 2$, $x_c = 10$, $y_c = 0$
Theorem 4.1. For a given \( n \) the maximum value \( \rho = \rho_{\text{max}} \) for which an intersection exists between the spheroid and a plane described by \( n \cdot u = \rho \) is given by

\[
\rho_{\text{max}} = \sqrt{B^2 + (A^2 - B^2)n_1^2} \quad (11)
\]

Proof. First, parameterize the spheroid with \( s \) and \( t \) such that

\[
\begin{align*}
  u &= A \cos s \\
  v &= B \sin s \cos t \\
  w &= B \sin s \sin t
\end{align*}
\quad (12)
\]

For any given \( s \) and \( t \), the normal vector \( n \) to the surface of the spheroid at this point can be found simply by taking the \( \text{grad} \) of the surface function:

\[
\begin{align*}
  F &= \frac{u^2}{A^2} + \frac{v^2}{B^2} + \frac{w^2}{B^2} \\
  \therefore \ \nabla F &= 2 \left( \frac{u}{A^2}, \frac{v}{B^2}, \frac{w}{B^2} \right) \\
  \therefore \ n &= \frac{1}{\sqrt{B^2 \cos^2 s + A^2 \sin^2 s}} \begin{pmatrix} B \cos s \\ A \sin s \cos t \\ A \sin s \sin t \end{pmatrix} \quad (13c)
\end{align*}
\]

We can re-arrange these to express \( s \) and \( t \) in terms of the components of \( n = (n_1, n_2, n_3) \). Then, to find \( \rho_{\text{max}} \) the equation for the plane \( n \cdot u = \rho \) is expanded, where \( u(s(n), t(n)) \) is a point on the spheroid. The resulting simplified expression is

\[
\rho_{\text{max}} = \sqrt{B^2 + (A^2 - B^2)n_1^2} \quad . \quad (14)
\]

Figure 5: Demonstrating the perspective effect on the bounding lines for an ellipse on the focal axis - \( a = 4, b = 2, x_c = 5 \).
Figure 6: Spinning the ellipse around its centre will give \( m_+ \) and \( m_- \) a range of values between that for a projected circle of radius \( a = 4 \) and of radius \( b = 2 \) in the same position - \( x_c = 20, y_c = 15 \).

We now introduce a point \( X_D \) which represents the focal point in the spheroid frame. Define the focal axis as a line passing through \( X_D \) with direction \( e \). Redefine the vector \( n = (n_1, n_2, n_3) \) to be orthogonal to \( e \) and a final base vector \( f = e \wedge n \) to complete the orthogonal set. Then for some \( \rho \),

\[
\begin{align*}
\mathbf{n} \cdot e &= 0 \\
\mathbf{n} \cdot X_D &= \rho 
\end{align*}
\]  

(15a) (15b)

The equation of the spheroid is given by

\[
u^T \Lambda^{-2} u = 1,
\]

(16)

where

\[
\Lambda \doteq \begin{pmatrix}
A & 0 & 0 \\
0 & B & 0 \\
0 & 0 & B
\end{pmatrix}
\]

(17)

and the intersecting curve is described by \( u \), where \( u \) satisfies both equations (16) and

\[
\mathbf{n} \cdot \mathbf{u} = \rho.
\]

(18)

To find \( u \) consider stretched axes such that \( \mathbf{Y} = \Lambda^{-1} \mathbf{u} \). We end up with a new expression
Figure 7: Demonstrating the perspective effect on the bounding lines for an ellipse off the focal axis - $a = 4$, $b = 2$, $x_c = 20$, $y_c = 15$. The vertical lines highlight the difference in $\theta$ between maximum and minimum values of $m_+$ and $m_-$. 

for the plane:

$$n' \cdot Y = \rho' ,$$  \hspace{1cm} (19a)

where $n' = \frac{1}{\rho_{\text{max}}} (A_{n_1}, B_{n_2}, B_{n_3})$ ,  \hspace{1cm} (19b)

$$\rho' = \frac{1}{\rho_{\text{max}}} \rho .$$  \hspace{1cm} (19c)

In this frame, the spheroid becomes a unit sphere, thus the intersection is a circle with radius $l = \sqrt{1 - \rho'^2}$. The variable $\rho_{\text{max}}$ is a key parameter. If $\rho$ is greater than $\rho_{\text{max}}$ then $\rho' > 1$ and $l$ is complex. This concurs with equation (14). We can parameterize the circle easily and describe the intersecting curve:

$$Y = \rho' n' + (l \cos \nu) \hat{c} + (l \sin \nu) \hat{d}$$  \hspace{1cm} (20)

In this stretched frame $\hat{c}$ and $\hat{d}$ are defined simply as two vectors making up an orthogonal base with $n'$. The rotational freedom about $n'$ which $\hat{c}$ and $\hat{d}$ have are superfluous when we revert back to the un-stretched frame:

$$u = \frac{\rho}{\sqrt{\rho_{\text{max}}}} (A^2 n_1 , B^2 n_2 , B^2 n_3 ) + \Lambda[ (l \cos \nu) \hat{c} + (l \sin \nu) \hat{d} ]$$  \hspace{1cm} (21)

After a little manipulation we can re-write the above equation in terms of the known
vectors $e$ and $f$. If we define \( \beta \triangleq A^2 n_1^2 + B^2 n_3^2 \), then

\[
\mathbf{u} = \Upsilon + \sigma e + \eta f,
\]

where \( \Upsilon \triangleq \frac{\rho}{\rho_{\text{max}}} \left( \begin{array}{c} A^2 n_1 \\ B^2 n_2 \\ B^2 n_3 \end{array} \right) \), \hfill (22a)

\[
\sigma \triangleq \frac{B \sqrt{\rho_{\text{max}}^2 - \rho^2}}{\rho_{\text{max}}^2 \sqrt{\beta}} \left( A f_2 \rho_{\text{max}} \cos \nu + (A^2 n_1 f_3 - B^2 n_3 f_1) \sin \nu \right), \hfill (22b)
\]

\[
\eta \triangleq \frac{B \sqrt{\rho_{\text{max}}^2 - \rho^2}}{\rho_{\text{max}}^2 \sqrt{\beta}} \left( -A e_2 \rho_{\text{max}} \cos \nu + (-A^2 e_3 n_1 + B^2 e_1 n_3) \sin \nu \right), \hfill (22c)
\]

The equation for \( \mathbf{u} \) describes the intersect curve in the spheroid’s coordinates. This intersection is an ellipse, the centre of which is given by \( \Upsilon \). This can be seen more explicitly when one calculates the radial distance \( \zeta(\nu) \) from \( \Upsilon \) to a point on this curve, to find

\[
\zeta(\nu)^2 = \frac{B^2 (\rho_{\text{max}}^2 - \rho^2)}{\rho_{\text{max}}^2 \beta} \left( \frac{1}{2} \Gamma_0 + \frac{1}{2} \Gamma_1 \cos 2\nu - A \rho_{\text{max}} \Gamma_2 \sin 2\nu \right), \hfill (23)
\]

where

\[
\Gamma_0 \triangleq \beta(A^2 + \rho_{\text{max}}^2) \hfill (24a)
\]

\[
\Gamma_1 \triangleq (A^2 - B^2)(\beta(1 - n_1^2) - 2n_1^2 n_2^2 A^2) \hfill (24b)
\]

\[
\Gamma_2 \triangleq (A^2 - B^2)n_1 n_2 n_3 \hfill (24c)
\]
We differentiate the parameterized distance of the curve from this point with respect to the parameter to find the semi-major axis $a_{\text{int}}$ and semi-minor axis $b_{\text{int}}$ of the intersecting ellipse. The orientation of the ellipse $\omega_{\text{int}}$ is defined as the angle of the semi-major axis away from $e$. It can be shown that the intersecting ellipse values are given by:

$$a_{\text{int}} = \frac{AB}{\rho_{\text{max}}} \sqrt{\rho_{\text{max}}^2 - \rho^2}$$  \hspace{1cm} (25a)

$$b_{\text{int}} = \frac{B}{\rho_{\text{max}}} \sqrt{\rho_{\text{max}}^2 - \rho^2}$$  \hspace{1cm} (25b)

$$\tan \omega_{\text{int}} = \frac{f_1}{e_1}$$  \hspace{1cm} (25c)

It is clear why $\rho$ must be less than $\rho_{\text{max}}$ in order for there to be real solutions to $a_{\text{int}}$ and $b_{\text{int}}$; if $\rho$ is greater than $\rho_{\text{max}}$, there is no intersection of the plane with the spheroid.

5 Perspective Projection of a Spheroid onto a Plane

The image plane is defined with an orthonormal vector as the focal axis of the camera system and a focal length $f$ - the perpendicular distance from the origin (the focal length then determines the magnification of the captured image). When considering the $x$ axis to be the focal axis and $x > 0$ to be the direction the camera is looking, a pin-hole camera has its image plane behind the focal point. By symmetry we can move the image plane in front of the focal point to obtain a more understandable correlation between the object and its projection.

In this section we derive the governing equations for the projection of the spheroid onto the image plane. Our goal is to find these expressions in the camera frame: The camera frame is defined relative to the spheroid frame as centred at the point $X_D$, with the first base vector $e$ which points along the focal axis. The other two axes are defined simply as being orthogonal to $e$: Later this rotational freedom is removed. The image plane is set at a dimensionless distance 1 from the focal point (the origin in the camera frame).

From section (4) the procedure now is to change the position of the origin to $X_D$ and redescribe the centre of the intersecting ellipse in the camera frame as $(x_{\text{int}}, y_{\text{int}})$. What we then have is a plane (given by it’s normal vector $n$) containing the origin and the vector $e$ which acts as a new $x$-axis. On this plane lies an ellipse whose semi-major axis is given by $a_{\text{int}}$, semi-minor axis given by $b_{\text{int}}$, centre $(x_{\text{int}}, y_{\text{int}})$ and orientation $\omega_{\text{int}}$. We can then use the results found in section (2) to project this ellipse onto the line $x = 1$ (in this new frame) and recover two gradient values $m_\pm$. Figure (9) shows this method in diagrammatic form.

The centre of the intersecting ellipse has coordinates $\Upsilon - X_D$ in the camera frame. We choose a new set of orthogonal base-vectors $(e, n, f)$ relative to the spheroid frame. In this intermediate coordinate system the centre of the ellipse has coordinates $(\lambda_1, \lambda_2, \lambda_3)$,
$\lambda_1 = \frac{\rho}{\rho_{\text{max}}} (B^2 (e \cdot n) - (A^2 - B^2) e_1 n_1) - e \cdot X_D$  \hspace{1cm} (26a)

$\lambda_2 = 0$  \hspace{1cm} (26b)

$\lambda_3 = \frac{\rho}{\rho_{\text{max}}} (A^2 - B^2) n_1 f_1 - f \cdot X_D$  \hspace{1cm} (26c)

(As expected the second component equals zero as the ellipse lies on this plane.) We also need to find expressions for $\cos 2 \omega_{\text{int}}$ and $\sin 2 \omega_{\text{int}}$ as this is the form they take in the 2D $m_{\pm}$ function (8). From equation (25), we can obtain a number of useful expressions:

$\sin 2 \omega_{\text{int}} = \frac{2e_1 f_1}{e_1^2 + f_1^2}$  \hspace{1cm} (27a)

$\cos 2 \omega_{\text{int}} = \frac{e_1^2 - f_1^2}{e_1^2 + f_1^2}$  \hspace{1cm} (27b)

$a_{\text{int}}^2 + b_{\text{int}}^2 = \frac{B^2}{\rho_{\text{max}}^4} (\rho_{\text{max}}^2 - \rho^2) (A^2 + \rho_{\text{max}}^2)$  \hspace{1cm} (27c)

$a_{\text{int}}^2 - b_{\text{int}}^2 = \frac{B^2}{\rho_{\text{max}}^4} (\rho_{\text{max}}^2 - \rho^2) (A^2 - \rho_{\text{max}}^2)$  \hspace{1cm} (27d)

$a_{\text{int}}^2 b_{\text{int}}^2 = \frac{A^2 B^4}{\rho_{\text{max}}^6} (\rho_{\text{max}}^2 - \rho^2)^2$  \hspace{1cm} (27e)

We now have all the required elements to substitute into equation (8), which can be
reduced to the following:

\[ m_{\pm} = \frac{C_1 \pm C_2}{D_1} \]  

(28)

where, writing \( X_D = (x_d, y_d, z_d) \),

\[ C_1 = -e_1 f_1 B^2 (A^2 - B^2) + B^2 \langle \vec{e} \cdot X_D \rangle (f \cdot X_D) - (A^2 - B^2) \langle \vec{e} \wedge X_D \rangle_1 (\vec{e} \wedge X_D)_1 \]  

(29)

\[ C_2 = B \sqrt{p_{\text{max}}^2 - \rho^2} \sqrt{B^2 (x_d^2 - A^2) + A^2 (y_d^2 + z_d^2)} \]  

(30)

\[ D_1 = B^2 \left( X_D^2 - (f \cdot X_D)^2 \right) - B^2 \left( A^2 - (A^2 - B^2) f_1^2 \right) + (A^2 - B^2) (f \wedge X_D)_1 \]  

(31)

We now formally define the camera frame: Define a new set of base vectors to be the camera frame axes relative to the spheroid frame as \((\vec{e}, \vec{i}_1, \vec{i}_2)\): This can be described as a rotation of the base vectors \((\vec{e}, \vec{n}, \vec{f})\) around \(\vec{e}\) by an angle \(\gamma\):

\[
\begin{align*}
\vec{i}_2 &= \vec{e} \wedge \vec{i}_1 \\
\vec{n} &= \vec{i}_1 \cos \gamma - \vec{i}_2 \sin \gamma \\
\gamma & \in [0, \pi]
\end{align*}
\]  

(32)

Let \(R_\gamma\) be defined as the rotation matrix mapping \((\vec{e}, \vec{n}, \vec{f})\) around \(\vec{e}\) to \((\vec{e}, \vec{i}_1, \vec{i}_2)\). Then define another rotation matrix \(R\) such that

\[ R : (\vec{e}, \vec{i}_1, \vec{i}_2) \rightarrow (u, v, w) \]  

(33)

(\(u, v, w\) were the original axes co-aligned with the spheroid’s principal axes). Note that \(\gamma\) need only range from 0 to \(\pi\), as \(m\) is not +ve definite. We choose \(R\) such that the first rotation is around the \(\vec{i}_2\) axis by an amount \(\theta\) (defined in the negative sense), the second rotation is around the ‘new’ \(\vec{i}_1\) axis by an amount \(\phi\) (also defined in the negative sense) and the final rotation is around the \(u\)-axis by an amount \(\psi\) - which has no effect, as the spheroid is symmetric around this axis. Thus we only need to consider two angles. The rotation matrix \(R\) relating the camera frame to the spheroid frame is hence given by

\[
\begin{pmatrix}
\cos \theta \cos \phi \\
\cos \theta \sin \phi \sin \psi + \sin \theta \cos \psi \\
-\cos \theta \sin \phi \cos \psi + \sin \theta \sin \psi
\end{pmatrix}
\begin{pmatrix}
-\sin \theta \cos \phi \\
-\sin \theta \sin \phi \sin \psi + \cos \theta \cos \psi \\
\sin \theta \sin \phi \cos \psi + \cos \theta \sin \psi
\end{pmatrix}
\begin{pmatrix}
\sin \phi \\
-\sin \phi \cos \psi + \cos \theta \sin \psi \\
\cos \phi \cos \psi
\end{pmatrix}
\]  

(34)

Note that \(p\), the vector pointing along the symmetric axis of the spheroid, is described in the camera frame by \(p_i = R(1, i)\). We also define \(q\) to be the position of the centre of the spheroid in the camera frame. This along with \(p\) is an important pairing for definitions:

\[
p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \phi \\ -\sin \theta \cos \phi \\ \sin \phi \end{pmatrix}
\]  

(35a)

\[
q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -R^T X_D
\]  

(35b)

\[
p \cdot p = 1 \\
r^2 = \frac{q \cdot q}{q \cdot q}
\]  

(35d)
We apply this 3-2-1 rotation matrix in (34) along with the $\gamma$ rotation matrix described by equation (32) to the elements in $C_1$, $C_2$ and $D_1$ (described in (28)), to write $e$, $n$ and $f$ in terms of $p$, $q$ and $\gamma$. The process described leaves us with an origin and set of base vectors $(e, \hat{e}_1, \hat{e}_2)$ which complete the camera frame. In this frame the first base-vector is the focal axis and for each angle $\gamma$ ($\gamma \in [0, \pi]$) measured from $\hat{e}_2$ towards $\hat{e}_1$ we have two functions $m_+(\gamma)$ and $m_-(\gamma)$. These functions can be described as:

$$m(\gamma)_+ = \frac{\delta_1 \cos \gamma + \delta_5 \sin \gamma \pm \sqrt{\delta_6 \delta_9 + \delta_7 \cos 2\gamma + \delta_8 \sin 2\gamma}}{\delta_1 + \delta_2 \cos 2\gamma + \delta_3 \sin 2\gamma}$$

$$\delta_i = \delta_i(A, B, q, p), \quad i = \{1, \ldots, 9\}.$$  

We later refer to equation (37) as the 'm-function'. The expressions in (37) are written in full below:

$$\delta_1 = \frac{1}{2} A^2 q_1^2 + \frac{1}{2} B^2 r^2 - \frac{1}{2} B^2 (A^2 + B^2) + (A^2 - B^2) \left[ \frac{1}{2} p_1^2 (r^2 - B^2) - (q \cdot p) q_1 p_1 \right]$$

$$\delta_2 = \frac{1}{2} B^2 (q_2^2 - q_3^2 - (A^2 - B^2)(p_2^2 - p_3^2)) + \frac{1}{2} (A^2 - B^2) \left[ (q \wedge p)_2^2 - (q \wedge p)_3^2 \right]$$

$$\delta_3 = -B^2 (q_2 q_3 - (A^2 - B^2) p_2 p_3) + (A^2 - B^2) (q \wedge p)_2 (q \wedge p)_3$$

$$\delta_4 = B^2 (q_1 q_3 - (A^2 - B^2) p_1 p_3) - (A^2 - B^2) (q \wedge p)_1 (q \wedge p)_3$$

$$\delta_5 = B^2 (q_1 q_2 - (A^2 - B^2) p_1 p_2) - (A^2 - B^2) (q \wedge p)_1 (q \wedge p)_2$$

$$\delta_6 = -\frac{1}{2} \left[ q_2^2 + q_3^2 - (A^2 - B^2)(p_2^2 + p_3^2) \right]$$

$$\delta_7 = -\frac{1}{2} \left[ q_2^2 - q_3^2 - (A^2 - B^2)(p_2^2 - p_3^2) \right]$$

$$\delta_8 = q_2 q_3 - (A^2 - B^2) p_2 p_3$$

$$\delta_9 = B^2 \left[ A^2 r^2 - A^2 B^2 - (A^2 - B^2)(p \cdot q)^2 \right]$$

The above equations will later be referred to as the $\delta$-equations.

These two $m$-functions describe the gradients of the bounding lines of the ellipse on the rotating plane. This ellipse is the intersection of the spheroid with the rotating plane. Figure (10) attempts to illustrate this more clearly.

If we now pan over all the values of $\gamma$ we collect a set of bounding gradients for all the cut-through’s of the spheroid (the gradient lines touching the spheroid along the extremal contour). As we defined the image plane to be a distance 1 along the focal axis, the gradient values are equivalent to bounding points on the image plane. These bounding points describe
Figure 10: The spheroid projection is calculated by finding the intersection of the spheroid with a rotating plane that contains the origin and the focal axis. This intersection is an ellipse which can then be projected towards the origin. The ellipse projection is defined by an \( m \) function derived in section (2).

the occluding contour on the projection plane - the outline of the perspective projection of the spheroid.

In the next section we create another \( m \)-function, created by expressing an ellipse on the image plane in terms of \( m \) and \( \gamma \). This second \( m \)-function is identical in construct to the first, thus we can deduce that the \( m \)-function derived in this section is that of an ellipse. More explicitly, the outline of the perspective projection of the spheroid is an ellipse.

6 Ellipse Parameters of a Perspective Projected Spheroid

This section approaches the perspective projection problem from the opposite end: The image plane ellipse. In order to make a connection between the spheroid projection and the image plane ellipse we must first describe them with the same parameters: \( \gamma \) and \( m \). Proceeding that we will then develop these connections to finally obtain a set of simplified expressions of the ellipse parameters in terms of the spheroid pose.

Define the camera frame to have axes \((x, y, z)\) so that the focal axis points along the \( x \)-axis and the \( m \)-equation given in (36) is valid. Define the image plane frame to be \( \{\mathbf{v} : \mathbf{v} \cdot (1, 0, 0) = 1\} \). We choose the image plane frame to be co-aligned with the camera frame with centre \((1, 0, 0)\) relative to the camera frame, but with just two components \((z, y)\) again aligned with the \( z \) and \( y \) axes of the camera frame. Thus the focal \( x \)-axis passes through the image plane frame at \((0,0)\).
Before we approach the final stage of this projection analysis, we define a new variable:

\[
\alpha = B^2 + (A^2 - B^2)p_1^2
\]  

(47)

The significance of this variable will be seen later, but observe \(\alpha\)’s physical properties first: Recall the equation of \(\rho_{\text{max}}(n)\) from theorem (4.1). Replacing \(n\) with \(p\) (and exercising the unit length property) \(\rho_{\text{max}}(p) = \alpha\). As theorem (4.1) is considering a spheroid with its axis of symmetry along the first / focal axis, then by replacing \(n\) with \(p\) this quantity is the square distance from the centre of the spheroid to the furthest (or nearest) extremity of the spheroid in the focal / \(x\) direction. This is shown in figure (11).

Figure 11: Physical interpretation of \(\alpha\).

Therefore should this value become greater than \(q_1^2\), the spheroid is intruding on the negative domain of the focal axis and may result in a hyperbolic projection onto the image plane. Should the value equal \(q_1^2\) the spheroid is touching the \(y - z\) plane in the camera frame, which will result in a parabolic projection onto the image plane.

Now let us consider an image plane ellipse. We describe the image plane ellipse in a similar construct to (36); in terms of an \(m_{\pm}\) function. Define the image plane ellipse to have centre \((Z_c,Y_c)\) (in the image plane frame), semi-major axis \(a\), semi-minor axis \(b\) and orientation \(\omega\) from the \(z\)-axis. This is shown in figure (12).

We find the intersecting points of the ellipse in much the same way that we found the cut-through on the spheroid; by changing the frame to match that of the ellipse. The exact method employed involves stretching the coordinate axes so that the ellipse is a circle which makes finding the intersect points far easier. We then stretch back to find the intersects for a general angle \(\gamma\). In the original image plane frame these points are exactly the desired \(m_{\pm}\) values. The expression is the same format as the projected spheroid (equation (36)).

**Theorem 6.1.** The equivalent \(m\)-function describing an ellipse in terms of an angle \(\gamma\) from the image plane \(z\)-axis and a distance \(m_{\pm}\) away from the origin, is given by:

\[
m(\gamma)_{\pm} = \frac{e_4 \cos \gamma + e_5 \sin \gamma \pm e_9 \sqrt{e_6 + e_7 \cos 2\gamma + e_8 \sin 2\gamma}}{e_1 + e_2 \cos 2\gamma + e_3 \sin 2\gamma}
\]  

(48)

\[
e_i = e_i(a, b, Z_c, Y_c, \omega) \quad , i = \{1, \ldots, 8\}
\]  

(49)
Figure 12: An ellipse on the image plane.

Proof. Define four frames which will be useful for this derivation:

$S_i$ The image plane frame.

$S_\gamma$ The frame centred at the image plane origin, but rotated around the focal axis by an amount $\gamma$. The $y$ and $z$ axes are then co-aligned with the $\mathbf{n}$ and $\mathbf{f}$ vectors defined in section (5).

$S_E$ The frame centred on $(Z_c,Y_c)$ oriented with the semi-major and minor axes of the image plane ellipse.

$S_E^*$ The frame centred at the image plane origin with axes oriented to match $S_E$.

In the $S_i$ frame the base vectors for $S_\gamma$ are given by $\mathbf{f}$ and $\mathbf{n}$:

$$\mathbf{f} = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} \quad \text{(50a)}$$

$$\mathbf{n} = \begin{pmatrix} -\sin \gamma \\ \cos \gamma \end{pmatrix} \quad \text{(50b)}$$

There are two rotation matrices to consider:

$$R_\gamma : S_i \rightarrow S_\gamma \Rightarrow R_\gamma = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \quad \text{(51)}$$

$$R_\omega : S_i \rightarrow S_E^* \Rightarrow R_\omega = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \quad \text{(52)}$$

Using these matrices and definitions, the image plane origin in $S_E$ is given by

$$\mathbf{x}_E = -\begin{pmatrix} Z_c \cos \omega + Y_c \sin \omega \\ -Z_c \sin \omega + Y_c \cos \omega \end{pmatrix} \quad \text{(53)}$$
We can also describe the vectors $f$ and $n$ in $S_E$ in a similar way. Staying in $S_E$ the ellipse is described by:

$$\mathbf{x}^T \Lambda \mathbf{x} = 1, \quad (54)$$

where

$$\Lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (55)$$

In a similar vein to the spheroid-plane intersection, we stretch the axes so that the ellipse becomes a circle:

$$\mathbf{x}' \div \Lambda^{-1} \mathbf{x} \quad (56)$$

When we consider the intersecting points of the circle with the sheered $f$ vector, this can now be expressed as

$$\mathbf{n}'_E \cdot \mathbf{x}' = \rho' \quad (57a)$$

$$\mathbf{n}'_E = \frac{1}{|\Lambda \mathbf{n}_E|} \Lambda \mathbf{n}_E \quad (57b)$$

$$\rho' = \frac{1}{|\Lambda \mathbf{n}_E|} \rho \quad (57c)$$

$$\mathbf{n}_E = R_\omega \mathbf{n} \quad (57d)$$

Now the intersecting points are easy to find. We now make the necessary transformations back until we return to the frame $S_i$. The intersecting points are given by the equation

$$\mathbf{X}_{int} = \begin{pmatrix} Z_{int} \\ Y_{int} \end{pmatrix}$$

$$= \begin{pmatrix} Z_c \\ Y_c \end{pmatrix} + \left( (\mathbf{n} \cdot \mathbf{x}_E) R_\omega^T \Lambda^2 R_\omega \mathbf{n} \pm ab \sqrt{|\Lambda \mathbf{n}_E|^2 - (\mathbf{n} \cdot \mathbf{x}_E)^2 f} \right) |\Lambda \mathbf{n}_E|^{-2} \quad (58)$$

(where $\mathbf{x}_E \div -R_\omega \begin{pmatrix} Z_c \\ Y_c \end{pmatrix}$). Once we’ve reached this stage one more transformation is required, from $S_i$ to $S_\gamma$: in this frame, the $n$ component will be zero and the other component is the $m$ function. Through straightforward expanding and simplifying we eventually reach the desired function given in equation (48). The $\varepsilon$ functions are given below.

$$\varepsilon_1 = (a^2 + b^2) \quad (59)$$

$$\varepsilon_2 = -(a^2 - b^2) \cos 2\omega \quad (60)$$

$$\varepsilon_3 = -(a^2 - b^2) \sin 2\omega \quad (61)$$

$$\varepsilon_4 = -(a^2 - b^2)(Z_c \cos 2\omega + Y_c \sin 2\omega) + (a^2 + b^2)Z_c \quad (62)$$

$$\varepsilon_5 = -(a^2 - b^2)(Z_c \sin 2\omega - Y_c \cos 2\omega) + (a^2 + b^2)Y_c \quad (63)$$
The above equations will later be referred to as the \( \varepsilon \)-equations.

Perhaps the most important relation in both the projective and reconstructive aspect of this construct is the corollary that now follows.

**Corollary 6.1. Inverse Equivalence**

\[
\begin{align*}
\frac{\delta_1}{\varepsilon_1} &= \frac{\delta_2}{\varepsilon_2} = \frac{\delta_3}{\varepsilon_3} = \frac{\delta_4}{\varepsilon_4} = \frac{\delta_5}{\varepsilon_5} \\
\frac{\delta_6}{\varepsilon_6} &= \frac{\delta_7}{\varepsilon_7} = \frac{\delta_8}{\varepsilon_8} = \left( \frac{\delta_1}{\varepsilon_1} \right)^2 \varepsilon_9 \\
\frac{\delta_9}{\varepsilon_9} &= \frac{\delta_9}{\varepsilon_9}
\end{align*}
\]

**Proof.** The main proof is contained in appendix A, however the basic outline is described: Starting at the relation \( m_{+/-,\delta} = m_{+/-,\varepsilon} \), by considering the sum and difference that these two equations yield we remove the \( +/− \) ambiguity. Then multiply up the fractions and re-write the trigonometric functions as \( \cos / \sin n\gamma \) which gives an orthogonal basis in \( \gamma \). Using this fact gives us a set of equations which can be described in matrix format:

\[
\begin{pmatrix}
-\frac{1}{2} \varepsilon_4 & -\frac{1}{2} \varepsilon_5 & \varepsilon_1 + \frac{1}{2} \varepsilon_2 & \frac{1}{2} \varepsilon_3 \\
-\frac{1}{2} \varepsilon_5 & -\frac{1}{2} \varepsilon_4 & \frac{1}{2} \varepsilon_3 & \varepsilon_1 + \frac{1}{2} \varepsilon_2 \\
\varepsilon_5 & \varepsilon_4 & -\varepsilon_3 & -\varepsilon_2 \\
-\varepsilon_4 & \varepsilon_5 & \varepsilon_2 & -\varepsilon_3
\end{pmatrix}
\begin{pmatrix}
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5
\end{pmatrix}
= 
\begin{pmatrix}
\varepsilon_4 \\
\varepsilon_5 \\
0 \\
0
\end{pmatrix}
\]

and likewise for the ‘difference’ equation. After inverting and simplifying the resulting equations the expressions in equation (68) fall out.

We now consider a change in variables. We can describe the image plane ellipse as a quadratic in two variables \( y \) and \( z \):

\[
A_1 z^2 + A_2 z y + A_3 y^2 + A_4 z + A_5 y + A_6 = 0
\]

where

\[
A_1 = \frac{1}{2} (a^2 + b^2) - \frac{1}{2} (a^2 - b^2) \cos 2\omega
\]

\[
A_2 = -(a^2 - b^2) \sin 2\omega
\]
can write the ellipse parameters in terms of the spheroid parameters by replacing $b$ and $\delta$ likewise for $A$.

Note, however, that these are all adequately given as ratios. Ergo using equation (68) we can write the ellipse parameters in terms of the spheroid parameters by replacing $\varepsilon_i$ with $\delta_i$ in the equations above and then simplifying - some extensively.

\begin{equation}
A_3 = \frac{1}{2}(a^2 + b^2) + \frac{1}{2}(a^2 - b^2)\cos 2\omega
\end{equation}

\begin{equation}
A_4 = -(a^2 + b^2)Z_c + (a^2 - b^2)Z_c\cos 2\omega + (a^2 - b^2)Y_c\sin 2\omega
\end{equation}

\begin{equation}
A_5 = -(a^2 + b^2)Y_c - (a^2 - b^2)Y_c\cos 2\omega + (a^2 - b^2)Z_c\sin 2\omega
\end{equation}

\begin{equation}
A_6 = \frac{1}{2}(a^2 + b^2)(Z_c^2 + Y_c^2) - \frac{1}{2}(a^2 - b^2)(Z_c^2 - Y_c^2)\cos 2\omega - (a^2 - b^2)Z_cY_c\sin 2\omega - a^2b^2
\end{equation}
\[ Y_c = \frac{q_1q_2 - (A^2 - B^2)p_1p_2}{q_1^2 - \alpha} \]  

\[ (a^2 - b^2)\cos 2\omega = \frac{[(A^2 - B^2)(p_3^2 - p_2^2) - (q_3^2 - q_2^2) + (q_1^2 - \alpha)(Z_c^2 - Y_c^2)]}{q_1^2 - \alpha} \]  

\[ (a^2 - b^2)\sin 2\omega = \frac{2[(A^2 - B^2)p_2p_3 - q_2q_3 + (q_1^2 - \alpha)Z_cY_c]}{q_1^2 - \alpha} \]  

\[ (a^2 + b^2) = \frac{(A^2 + B^2) - (A^2 - B^2)p_1^2 - (q_2^2 + q_3^2) + (q_1^2 - \alpha)(Z_c^2 + Y_c^2)}{q_1^2 - \alpha} \]  

The choice of combining \((a^2 - b^2)\cos 2\omega\), \((a^2 - b^2)\sin 2\omega\) and \((a^2 + b^2)\) is because their expressions in terms of a mixture of the spheroid parameters and other ellipse parameters are much simpler than they are individually. A trivial set of steps is all that is required to convert these expressions to single expressions for \(a\), \(b\) and \(\tan 2\omega\). It is also in this form that spheroid reconstruction becomes simpler, however this aspect is not considered in this paper.

One can see in each of these expressions the relevance of \(q_1^2 - \alpha\) and how it plays a large part in determining the type of conic the projection yields. However the true value of this expression is seen by linking the approach we have explored back to the approach of homogeneous coordinates and calculating polar planes (see Wylie [15]). The position \(q\) and orientation \(p\) of the spheroid lie in a subspace of \(\mathbb{R}^3 \times \mathbb{R}^3\) defined by the unit constraint on \(p\), where this subspace describes all possible spheroids. Likewise all possible image plane ellipses can be described in a 5-dimensional space by the variables given in equations (84) to (88) above. The mapping of the spheroid space to the image plane ellipse space is encapsulated by quotients of quadratic combinations of the spheroid parameters with the common denominator given by \(q_1^2 - \alpha\). This is analogous to dividing by the 4th homogeneous coordinate in standard projections of \(\mathbb{R}^4\). Moreover, we can collapse the spheroid down to a point by taking \(A = B = 0\) in equations (84) to (88) to obtain the traditional homogeneous coordinate description of a point projection: As \(\alpha \to 0\),

\[ Z_c = \frac{q_3}{q_1} \]  

\[ Y_c = \frac{q_2}{q_1} \]  

\[ (a^2 - b^2)\cos 2\omega = 0 \]  

\[ (a^2 - b^2)\sin 2\omega = 0 \]  

\[ (a^2 + b^2) = 0 \]  

Equations (84) to (88) describe the projection mapping from the spheroid space to the space of image plane ellipses and shows that there is an equivalence between the homogeneous coordinate method for the projection of points in \(\mathbb{R}^4\) and the projection of spheroids in \(\mathbb{R}^3 \times \mathbb{R}^3\).
For completeness, it is now possible to return back to the $\delta_i - \varepsilon_i$ equations (equations (38)-(46) and (59)-(67)) and find the coefficient linking them all:

Define $\kappa \overset{\triangle}{=} \frac{1}{2}(q_{12}^2 - \alpha)^2$

Then $\delta_i = \kappa \varepsilon_i$, $i = 1, \ldots, 5$
\[ \delta_j = \sqrt{2} \kappa \varepsilon_j, \quad j = 6, 7, 8 \]
\[ \delta_j \delta_9 = \kappa^2 \varepsilon_j \varepsilon_9, \quad j = 6, 7, 8 \]

(90)

Corollary 6.2. Image Plane Axial Symmetry

The image plane $Y$ and $Z$ axes are arbitrary: A rotation of the spheroid parameters $q$ and $p$ by an angle $\phi$ about the focal axis will result in an equal rotation of the image plane ellipse parameters $Z_c, Y_c$ and $\omega$.

Proof. Define a rotation about the focal axis:

$$ R_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} $$

(91)

The rotation $R_\phi$ takes the camera frame to a rotated frame we define as $S_\phi$. Finally denote parameters described in $S_\phi$ with a bracketed $\phi$ subscript.

$$ q_{(\phi)} = R_\phi^T q $$

(92a)

$$ p_{(\phi)} = R_\phi^T p $$

(92b)

Hence $q_{(\phi)} = q$.

By expanding out terms for $Z_{c(\phi)}$ and $Y_{c(\phi)}$ (from equations (84) and (85)) we can write the ellipse centre in terms of the original camera frame coordinates:

$$ Z_{c(\phi)} = \frac{q_{1(\phi)}q_{3(\phi)} - (A^2 - B^2)p_{1(\phi)}p_{3(\phi)}}{q_{1(\phi)}^2 - B^2 - (A^2 - B^2)p_{1(\phi)}^2} $$
\[ = \frac{q_1(-q_2 \sin \phi + q_3 \cos \phi) - (A^2 - B^2)p_1(-p_2 \sin \phi + p_3 \cos \phi)}{q_1^2 - B^2 - (A^2 - B^2)p_1^2} \]
\[ = -Y_c \sin \phi + Z_c \cos \phi \]

(93a)

$$ Y_{c(\phi)} = \frac{q_{1(\phi)}q_{2(\phi)} - (A^2 - B^2)p_{1(\phi)}p_{2(\phi)}}{q_{1(\phi)}^2 - B^2 - (A^2 - B^2)p_{1(\phi)}^2} $$
\[ = \frac{q_1(q_2 \cos \phi + q_3 \sin \phi) - (A^2 - B^2)p_1(p_2 \cos \phi + p_3 \sin \phi)}{(q_1^2 - B^2 - (A^2 - B^2)p_1^2} \]
\[ = Y_c \cos \phi + Z_c \sin \phi \]

(94a)

Therefore

$$ \begin{pmatrix} 1 \\ Y_{c(\phi)} \\ Z_{c(\phi)} \end{pmatrix} = R_\phi^T \begin{pmatrix} 1 \\ Y_c \\ Z_c \end{pmatrix} $$

(95)
As \( q_1(\phi) = q_1 \) and \( r_\phi = r \) then \( q_2(\phi)^2 + q_3(\phi)^2 = q_2^2 + q_3^2 \), with equivalent relations for \( p_\phi \) and \( (1, Y_{c(\phi)}, Z_{c(\phi)}) \). By considering equation (88), showing that \( a(\phi)^2 + b_1(\phi)^2 = a^2 + b^2 \) is now trivial with these relations.

Finally we look at the cosine and sine equations given in equations (86) and (87). Consider the following relation:

\[ p_{2(\phi)} p_{3(\phi)} = \frac{1}{2}(p_3^2 - p_2^2) \sin 2\phi + p_2 p_3 \cos 2\phi \]

(96)

The equivalent relations hold for \( q_{2(\phi)} \) and \( (1, Y_{c(\phi)}, Z_{c(\phi)}) \). As such,

\[
(a_\phi)^2 - b_\phi^2) \sin 2\omega_\phi = (a^2 - b^2) \cos 2\omega \sin 2\phi + (a^2 - b^2) \sin 2\omega \cos 2\phi \quad (97a)
\]

\[
\rightarrow (a_\phi)^2 - b_\phi^2) \sin 2\omega_\phi = (a^2 - b^2) \sin 2(\omega + \phi) \quad (97b)
\]

By repeating a similar process for the expressions found in equation (86), we obtain:

\[
(a_\phi)^2 - b_\phi^2) \cos 2\omega_\phi = (a^2 - b^2) \cos 2\omega \cos 2\phi - (a^2 - b^2) \sin 2\omega \sin 2\phi \quad (98a)
\]

\[
\rightarrow (a_\phi)^2 - b_\phi^2) \cos 2\omega_\phi = (a^2 - b^2) \cos 2(\omega + \phi) \quad (98b)
\]

Taking the sum of the squares of equations (97) and (98) yields how \( a^2 - b^2 \) is invariant, thus combined with the \( a^2 + b^2 \) invariance the individual terms are unaffected by rotation. Taking the quotient of these equations shows the effect on \( \omega \):

\[
\tan 2\omega_\phi = \tan 2(\omega + \phi) \quad (99)
\]

Therefore the system is rotationally invariant about the focal axis.

7 Spheroid Perspective Projection Results

The nature of perspective projections has been discussed in section (3) and now we see this effect for a spheroid projected onto an image plane. Recall that the distances are normalized such that the image plane lies at a distance 1 from the focal point. The conversion from ‘real space’ to ‘unit space’ is obtained simply by scaling \( q \), \( A \) and \( B \) by \( 1/f \), where \( f \) is the camera’s focal length. To translate the image plane ellipse parameters back to ‘real space’ just requires multiplying \( Z_c, Y_c, a \) and \( b \) by \( f \). The angles \( p \) and \( \omega \) are unchanged.

We first consider the simple example of a sphere projection, of size \( A = B = 4 \), placed at \( q_1 = 20 \). By varying \( q_2 \) and \( q_3 \) from \(-45\) to \(+45\) we obtain the perspective stretching in figure (13).

There is an axial symmetry about the focal axis which can be seen in the sphere projection scenario and this symmetry holds true for all spheroids due to the nature of the projection method. This was more formally stated in corollary (6.2).

Because of the bias in \( \omega \) that is observed in figure (13) due to perspective stretching it is interesting to ask if there are limitations on the range of \( \omega \) for a general spheroid in a general position. For given values of \( A, B, Z_c, Y_c, a \) and \( b \) what are the possible values which \( \omega \) can take? We can quickly determine an estimate of this restriction by taking advantage
of the rotational symmetry about the focal axis proved in corollary (6.2), determining $\omega$ as a function of $p$ and numerically searching through $p$ for the greatest variance in $\omega$ away from $\omega_0$, where $\omega_0 = \tan^{-1} Y_c/Z_c$. The value of $(\omega - \omega_0)$ can take values between $\pm0^\circ$ to $\pm90^\circ$ so we can numerically generate a graph of the range of $(\omega - \omega_0)$ (from $0^\circ$ to $180^\circ$) against its radial distance on the image plane $R_c = \sqrt{Z_c^2 + Y_c^2}$. This is shown in figure (14): $q_1$ is fixed at a distance of 50 and then moved orthogonal to the focal axis to generate a typical ellipse projection with a given value of $R_c$. The values of $R_c$, $a$ and $b$ are then fixed and the range of $(\omega - \omega_0)$ is then assessed. Figure (14) shows the effect on this range from making the spheroid more prolate. For an extreme wide-angle lens camera the focal length can be 6mm with a lens radius of up to 35mm, which is equivalent to seeing points at a radial distance of $R_c = 6$ on the image plane. The semi-major axis of the spheroid goes up to twice the semi-minor axis in this figure and though we should expect some numerical inaccuracy figure (14) shows a limit to the ranges of $(\omega - \omega_0)$. This highlights a fundamental difference between perspective projections and orthographic projections: for an orthographic projection no such constraint exists. One can also consider the extreme cases: when $A \rightarrow B$ there is no freedom on $\omega$ and no possible deviation away from $\omega_0$; when $B \rightarrow 0$ there is no constraint on what $\omega$ can be. Another interesting approach to this problem is by considering the limitations of $a$ against $b$ for a given spheroid eccentricity, however this is best answered by first finding a reconstruction algorithm.

A demonstration of the theory is shown with figures (15) and (16): A spheroid of semi-major axis $A = 8$ and semi-minor axis $B = 3$ is given a centre position $q = [30, 3, 5]$ and orientation $p = [0.7348, -0.5950, 0.3256]$ (corresponding to $\theta = 39^\circ$ and $\phi = 19^\circ$), shown in figure (15). A picture is then generated in figure (16) using the ray-tracing program POV-Ray (trademark of Persistence of Vision Raytracer Pty. Ltd.) which approximates a perspective

Figure 13: Illustrating the perspective projection of a sphere with radius $A = 4$ and focal distance $q_1 = 20$, for varying $q_2$ and $q_3$. 

A demonstration of the theory is shown with figures (15) and (16): A spheroid of semi-major axis $A = 8$ and semi-minor axis $B = 3$ is given a centre position $q = [30, 3, 5]$ and orientation $p = [0.7348, -0.5950, 0.3256]$ (corresponding to $\theta = 39^\circ$ and $\phi = 19^\circ$), shown in figure (15). A picture is then generated in figure (16) using the ray-tracing program POV-Ray (trademark of Persistence of Vision Raytracer Pty. Ltd.) which approximates a perspective
Figure 14: Illustrating the range of possible values for \((\omega - \omega_0)\) against the radial distance \(R_c\) of the ellipse centre \((Z_c, Y_c)\) from the origin. The upper limit gives a constraint upon the possible angles for a given \(R_c\) in order for reconstruction to be possible. \(q_1 = 50\), \(B = 5\) and \(A\) varies from \(1.01 \times B\) (light grey line) to \(2B\) (black line).

projection. The analytic projection of this spheroid is calculated (using equations (84)-(88)) and scaled so as to match the camera specifications of the simulation: lengths are multiplied by the ratio of the number of pixels to the unit distance on the image plane and the centres shifted to match the frame of the camera. The resulting ellipse boundary is also shown and identically fits around the spheroid projection.

8 Conclusion

The methodology and governing equations for a spheroid perspective projection onto a plane has been derived, finishing with an exact description of the image plane ellipse parameters in terms of the spheroid parameters. Application of this theory is simple for computer simulations and this is demonstrated in section (7).

Distortions occur in perspective projections and these attributes have been discussed in sections (3) and (7). It is identified that there are constraints on the possible values of the ellipse orientation for a given spheroid dimensions, a fixed ellipse centre and a fixed ellipse semi-major and semi-minor axes. With wide-angle cameras these constraints play a larger role and will have to be considered if the application involves reconstruction of the spheroid from its perspective projection. This raises the obvious question of how the other ellipse parameters are constrained and whether a set of analytic expressions can be found to describe these.
Figure 15: An example scenario with a spheroid of semi-major-axis 8 and semi-minor-axis 3 in the image space prior to projection.

Figure 16: Comparison between a spheroid perspective projection by ray-tracing and its analytical ellipse projection (shown by the occluding contour).
A Proof of Inverse Equivalence Corollary

This appendix looks at an expanded description of the Inverse Equivalence Corollary. The methodology is straightforward but analytically intense in places. In these circumstances only the theory is described.

Corollary A.1. Inverse Equivalence

\[
\begin{align*}
\frac{\delta_1}{\varepsilon_1} &= \frac{\delta_2}{\varepsilon_2} = \frac{\delta_3}{\varepsilon_3} = \frac{\delta_4}{\varepsilon_4} = \frac{\delta_5}{\varepsilon_5} \quad (100a) \\
\frac{\delta_6}{\varepsilon_6} &= \frac{\delta_7}{\varepsilon_7} = \frac{\delta_8}{\varepsilon_8} = \left( \frac{\delta_1}{\varepsilon_1} \right)^2 \frac{\varepsilon_9}{\delta_9} \quad (100b)
\end{align*}
\]

Proof. First define some new variables:

\[
\begin{align*}
E_1 &= \delta_4 \cos \gamma + \delta_5 \sin \gamma \\
E_2 &= \delta_6 + \delta_7 \cos 2\gamma + \delta_8 \sin 2\gamma \\
D_1 &= \delta_1 + \delta_2 \cos 2\gamma + \delta_3 \sin 2\gamma \\
E_3 &= \varepsilon_4 \cos \gamma + \varepsilon_5 \sin \gamma \\
E_4 &= \varepsilon_6 + \varepsilon_7 \cos 2\gamma + \varepsilon_8 \sin 2\gamma \\
D_2 &= \varepsilon_1 + \varepsilon_2 \cos 2\gamma + \varepsilon_3 \sin 2\gamma
\end{align*}
\]

Then

\[
\begin{align*}
m_\pm(\gamma, \delta_i) &= \frac{E_1 \pm \sqrt{\delta_9} \sqrt{E_2}}{D_1} \quad (102a) \\
m_\pm(\gamma, \varepsilon_i) &= \frac{E_3 \pm \sqrt{\varepsilon_9} \sqrt{E_4}}{D_2} \quad (102b)
\end{align*}
\]

We start with the relation \( m_{+/-}(\gamma, \delta_i) = m_{+/-}(\gamma, \varepsilon_i) \) for all \( \gamma \). By considering the sum and difference that these two equations yield we remove the +/- ambiguity:

\[
\begin{align*}
\frac{E_1}{D_1} &= \frac{E_3}{D_2} \quad (103a) \\
\frac{\delta_9 E_2}{D_1^2} &= \frac{\varepsilon_9 E_4}{D_2^2} \quad (103b)
\end{align*}
\]

Now multiply up the fractions and re-write the trigonometric functions as \( \cos n\gamma \) and \( \sin n\gamma \) which gives an orthogonal basis in \( \gamma \). For equation (103b) this is a complicated expression but has a similar structure to equation (103a), written here:

\[
\begin{align*}
&\left[ \delta_4 \varepsilon_1 - \delta_1 \varepsilon_4 + \frac{1}{2} \delta_5 \varepsilon_3 - \frac{1}{2} \delta_3 \varepsilon_5 + \frac{1}{2} \delta_4 \varepsilon_2 - \frac{1}{2} \delta_2 \varepsilon_4 - \frac{1}{2} \delta_3 \varepsilon_5 - \frac{1}{2} \delta_4 \varepsilon_2 \right] \cos \gamma \\
+ &\left[ \delta_5 \varepsilon_1 - \delta_1 \varepsilon_5 - \frac{1}{2} \delta_3 \varepsilon_3 + \frac{1}{2} \delta_5 \varepsilon_2 + \frac{1}{2} \delta_5 \varepsilon_4 - \frac{1}{2} \delta_3 \varepsilon_4 + \frac{1}{2} \delta_4 \varepsilon_3 \right] \sin \gamma \\
+ &\left[ \delta_4 \varepsilon_2 - \delta_2 \varepsilon_4 + \delta_3 \varepsilon_5 - \delta_5 \varepsilon_3 \right] \sin 3\gamma \\
+ &\frac{1}{2} \left[ \delta_2 \varepsilon_5 - \delta_5 \varepsilon_2 + \delta_4 \varepsilon_4 - \delta_1 \varepsilon_5 \right] \cos 3\gamma = 0
\end{align*}
\]
These equations must be satisfied for all $\gamma$ and as such the coefficients of $\cos n\gamma$ and $\sin n\gamma$ must be zero. Using this fact gives us a set of equations which can be described in matrix format:

\[
\begin{pmatrix}
-\frac{1}{5}\varepsilon_4 & -\frac{1}{5}\varepsilon_5 & \varepsilon_1 + \frac{1}{2}\varepsilon_2 & \frac{1}{2}\varepsilon_3 \\
-\frac{5}{2}\varepsilon_5 & -\frac{5}{2}\varepsilon_4 & \frac{7}{2}\varepsilon_3 & \varepsilon_1 + \frac{1}{2}\varepsilon_2 \\
\varepsilon_5 & \varepsilon_4 & -\varepsilon_3 & -\varepsilon_2 \\
-\varepsilon_4 & \varepsilon_5 & \varepsilon_2 & -\varepsilon_3
\end{pmatrix}
\begin{pmatrix}
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5
\end{pmatrix}
= 
\begin{pmatrix}
\varepsilon_4 \\
\varepsilon_5 \\
0 \\
0
\end{pmatrix}
\delta_1
\]

(105)

Refer to the above matrix on the left hand side as $F$. Then we have:

\[
\begin{pmatrix}
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5
\end{pmatrix}
= 
\frac{G}{\det F}
\begin{pmatrix}
\varepsilon_4 \\
\varepsilon_5 \\
0 \\
0
\end{pmatrix}
\delta_1
\]

(106)

where

\[
G = \frac{F^{-1} \det F}{\varepsilon}
\]

(107a)

\[
G_{11} = (\varepsilon_1 - \varepsilon_2)(\varepsilon_2\varepsilon_4 + \varepsilon_3\varepsilon_5)
\]

(107b)

\[
G_{12} = (\varepsilon_1 + \varepsilon_2)(-\varepsilon_3\varepsilon_4 + \varepsilon_2\varepsilon_5)
\]

(107c)

\[
G_{13} = (\varepsilon_1 + \varepsilon_2)(\varepsilon_1\varepsilon_5 + \frac{5}{2}\varepsilon_2\varepsilon_5 - \frac{7}{2}\varepsilon_3\varepsilon_4)
\]

(107d)

\[
G_{14} = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)(\varepsilon_3\varepsilon_5 - 4\varepsilon_1\varepsilon_2)
\]

(107e)

\[
G_{21} = \varepsilon_3\varepsilon_4(\varepsilon_1 - \varepsilon_2) - \varepsilon_5(\varepsilon_1\varepsilon_2 + \varepsilon_3^2)
\]

(107f)

\[
G_{22} = (\varepsilon_1 + \varepsilon_2)\varepsilon_3\varepsilon_5 + (\varepsilon_1\varepsilon_2 - \varepsilon_3^2)\varepsilon_4
\]

(107g)

\[
G_{23} = \varepsilon_4(\varepsilon_1^2 - \frac{5}{2}\varepsilon_1\varepsilon_2) - \frac{7}{2}\varepsilon_3\varepsilon_4 + \frac{5}{2}\varepsilon_3\varepsilon_5(\varepsilon_1 + \varepsilon_2)
\]

(107h)

\[
G_{24} = \frac{1}{2}[(\varepsilon_2 - \varepsilon_1)\varepsilon_3\varepsilon_4 + \varepsilon_1\varepsilon_5(\varepsilon_1 + \varepsilon_2) + \varepsilon_5(\varepsilon_1^2 - \varepsilon_3^2)]
\]

(107i)

\[
G_{31} = \varepsilon_4^2(\varepsilon_1 - \varepsilon_2) + \varepsilon_5(\varepsilon_1\varepsilon_5 - \varepsilon_3\varepsilon_4)
\]

(107j)

\[
G_{32} = \varepsilon_4(\varepsilon_2\varepsilon_5 - \varepsilon_3\varepsilon_4)
\]

(107k)

\[
G_{33} = \varepsilon_4(\varepsilon_5(\varepsilon_1 + \frac{5}{2}\varepsilon_2) - \frac{7}{2}\varepsilon_3\varepsilon_4)
\]

(107l)

\[
G_{34} = \frac{1}{2}[(\varepsilon_2 - \varepsilon_1)\varepsilon_4^2 + \varepsilon_5(\varepsilon_1\varepsilon_5 - \varepsilon_3\varepsilon_4)]
\]

(107m)

\[
G_{41} = -\varepsilon_5(\varepsilon_2\varepsilon_4 + \varepsilon_3\varepsilon_5)
\]

(107n)

\[
G_{42} = \varepsilon_5^2(\varepsilon_1 + \varepsilon_2) + \varepsilon_4(\varepsilon_1\varepsilon_4 - \varepsilon_3\varepsilon_5)
\]

(107o)

\[
G_{43} = \frac{1}{2}[5\varepsilon_5^2(\varepsilon_1 + \varepsilon_2) + 7\varepsilon_1\varepsilon_4^2 - 7\varepsilon_3\varepsilon_4\varepsilon_5]
\]

(107p)

\[
G_{44} = \varepsilon_4\varepsilon_5(\varepsilon_1 + \frac{1}{2}\varepsilon_2) - \frac{1}{2}\varepsilon_3\varepsilon_5^2
\]

(107q)

\[
\det F = \varepsilon_1[\varepsilon_1(\varepsilon_4^2 + \varepsilon_5^2) + \varepsilon_2(-\varepsilon_4^2 + \varepsilon_5^2) - 2\varepsilon_3\varepsilon_4\varepsilon_5]
\]

(107r)
Now consider each line in (106) and simplify the expression of $\varepsilon$’s. The first line simplifies to:

$$\frac{\delta_2}{\varepsilon_2} = \frac{\delta_1}{\varepsilon_1}$$  \hspace{1cm} (108)

Likewise for the other lines similar relations are deduced. This leads to the relation given in equation (100a).

To recover equation (100b), the same process is repeated from equation (103b) for the higher indexed $\delta$’s, but with one exception; use the results from equation (100a) to write $\delta_2$, $\delta_3$, $\delta_4$ and $\delta_5$ in terms of $\delta_1$, to obtain:

$$
\begin{pmatrix}
\frac{\varepsilon_1^2 + \frac{1}{2}\varepsilon_2^2 + \frac{1}{2}\varepsilon_3^2}{2\varepsilon_1\varepsilon_2} & \frac{\varepsilon_1\varepsilon_2}{2\varepsilon_1\varepsilon_3} & \frac{\varepsilon_1\varepsilon_3}{2\varepsilon_2\varepsilon_3} & \varepsilon_1^2 + \varepsilon_2^2
\end{pmatrix}
\begin{pmatrix}
\frac{\delta_6\delta_9}{\delta_7\delta_9} & \frac{\delta_7\delta_9}{\delta_8\delta_9}
\end{pmatrix}
= 
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
\varepsilon_9 \delta_1^2$$

(109)

where

$$\nu_1 \doteq \frac{\varepsilon_7}{\varepsilon_1} + \frac{\varepsilon_8}{\varepsilon_1} + \left(1 + \frac{1}{2} \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 + \frac{1}{2} \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^2\right) \varepsilon_6$$

(110a)

$$\nu_2 \doteq 2 \frac{\varepsilon_2}{\varepsilon_1} \left(\frac{\varepsilon_6 + \varepsilon_8}{\varepsilon_1}\right) + \left(1 + \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^2\right) \varepsilon_7$$

(110b)

$$\nu_3 \doteq 2 \frac{\varepsilon_3}{\varepsilon_1} \left(\frac{\varepsilon_6 + \varepsilon_7}{\varepsilon_1}\right) + \left(1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) \varepsilon_8$$

(110c)

From here, repeating the same process as the lower indexed terms we obtain a relation from each line. The first reads:

$$\frac{\delta_6}{\varepsilon_6} = \left(\frac{\delta_1}{\varepsilon_1}\right)^2 \frac{\varepsilon_9}{\delta_9}$$

(111)

The remaining equations have a similar construct, whereupon we can finally deduce equation (100b). 

\[\square\]

References


REFERENCES


